Regarding the 1D heat equation, we define

$$
\begin{aligned}
Q_T =&(x,t) : 0 < x < L, 0 < t \leqslant T,\\
\partial_p Q_T =&\{(x,t) : x = 0 \text{or} L, 0 < t \leqslant T\} \cup \{(x,0) : 0 \leqslant x \leqslant L\},\\
\overline{Q_T} =&Q_T \cup \partial_p Q_T.
\end{aligned}
$$

In the higher dimensions,

$$
\begin{aligned}
Q_T =&(\vec{x},t) : \vec{x} \in \Omega, 0 < t \leqslant T,\\
\partial_p Q_T =&\{(\vec{x},t) : \vec{x} \in \partial\Omega, 0 < t \leqslant T\} \cup \{(\vec{x},0) : \vec{x} \in \overline{\Omega} = \Omega \cup \partial\Omega\},\\
\overline{Q_T} =&Q_T \cup \partial_p Q_T.
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^n$ is a connected bounded open set with smooth boundary $\partial\Omega$.

In addition, we denote the outward pointing unit normal vector of $\partial\Omega$ by ν.

Moreover, regarding a function $u(\vec{x},t) = u(x_1, x_2, \cdots, x_n, t)$ we define

$$
\begin{aligned}
\nabla u =&(u_1, \cdots, u_n) = (\partial_{x_1} u, \cdots, \partial_{x_n} u),\\
\Delta u =&u_{11} + \cdots + u_{nn} = \sum_{i=1}^{n} \partial_{x_i}^2 u.
\end{aligned}
$$

Definition 1. Suppose that a function satisfies $u_t \leqslant \Delta u$ [resp. $u_t \geqslant \Delta u$] in $\overline{Q_T}$. Then, we call u a subsolution [resp. supersolution] to the heat equation.

Theorem 2 (Weak maximum principle)**.** *A subsolution [resp. supersolution] u to the heat equation satisfies*

$$
\max_{\overline{Q_T}} u \leqslant \max_{\partial_p Q_T} u, \qquad [\textit{resp.} \min_{\overline{Q_T}} u \leqslant \min_{\partial_p Q_T} u]. \tag{1}
$$

See the textbook Theorem 2.4 for the proof.

Theorem 3 (Comparison principle)**.** *Suppse that a subsolution u and a supersolution v to the heat equation satisfy $u \leqslant v$ on $\partial_p Q_T$. Then, $u \leqslant v$ holds on Q_T.*

See the textbook Corollary 2.5 for the proof.

Example (Interior estimate for time)**.** Let $u(x,t)$ be the solution to the 1D Cauchy-Neumann problem with $-u_x(0,t) = u_x(L,t) = 0$ and $u(x,0) = g(x)$. Then, the following holds

$$u_x^2 \leqslant \tfrac{1}{2}(1 + \tfrac{1}{t}) \max_{\overline{Q_T}} u^2.$$

Note that combining with the problem #1 in pset 2, one can obtain

$$u_x^2 \leqslant \tfrac{1}{2}(1 + \tfrac{1}{t}) \max_{0 \leqslant x \leqslant L} g^2.$$

Proof. We begin by defining a function $w = \frac{t}{2(t+1)}|u_x|^2 + \frac{1}{4}u^2$. Differentiating w yields

$$\begin{aligned} w_t &= \tfrac{1}{2(t+1)^2} u_x^2 + \frac{t}{t+1} u_x u_{xt} + \tfrac{1}{2} u u_t \leqslant \tfrac{1}{2} u_x^2 + \frac{t}{t+1} u_x u_{xt} + \tfrac{1}{2} u u_t, \\ w_{xx} &= \tfrac{1}{t+1} u_{xx}^2 + \tfrac{1}{t+1} u_x u_{xxx} + \tfrac{1}{2} u_x^2 + \tfrac{u}{u}_{xx} \geqslant \tfrac{1}{2} u_x^2 + \tfrac{1}{t+1} u_x u_{xxx} + \tfrac{1}{2} u u_{xx}, \end{aligned}$$

namely $w_t \leqslant w_{xx}$. Hence, the maximum principle yields,

$$\tfrac{t}{2(t+1)} u_x^2 \leqslant \max_{\overline{Q_T}} w = \max_{\partial_p Q_T} w = \max\{w(x,t) : x = 0, x = L, \text{or } t = 0\}.$$

By the Neumann condition, we have $\frac{t}{2(t+1)} u_x^2 = \frac{t}{2(t+1)} \cdot 0 = 0$ where $x = 0$ or $x = L$. In addition, if $t = 0$, then $\frac{t}{2(t+1)} u_x^2 = \frac{0}{2(0+1)} u_x^2 = 0$. Namely, on the parabolic boundary $\partial_p Q_T$ the following holds.

$$w = \tfrac{1}{4} u^2 \leqslant \tfrac{1}{4} \max_{\overline{Q_T}} u^2.$$

Therefore,

$$u_x^2 \leqslant \tfrac{1}{2}(1 + \tfrac{1}{t}) \max_{\overline{Q_T}} u^2.$$

□

The result in the example above implies that slopes u_x of the solution can be controlled short time later even if the initial data g has extremely high slopes $|g_x|$.

Example (Barriers)**.** Let $u(\vec{x},t)$ be the solution to the nD Cauchy-Dirichlet problem with $u(\vec{x},t) = 0$ for $\vec{x} \in \partial\Omega$ and $u(x,0) = g(x)$ for $\vec{x} \in \overline{\Omega}$. In addition, Ω is a convex set. Then, the following holds

$$|\nabla u|^2 \leqslant \max_{\overline{Q_T}} |\nabla g|^2.$$

See the bonus problem in pset 2 to work on non-convex domains.

Proof. We define $w = |\nabla u|^2$. Then,

$$\begin{aligned} w_t =&2\sum_{i=1}^n u_i u_{it} = 2\sum_{i=1}^n u_i \partial_i \Delta u = 2\sum_{i=1}^n u_i \partial_i \Big(\sum_{j=1}^n u_{jj}\Big) = 2\sum_{i,j=1}^n u_i u_{ijj}, \\ w_{jj} =&2\sum_{i=1}(u_{ij}^2 + u_i u_{ijj}) \geqslant 2\sum_{i=1}^n u_i u_{ijj}. \end{aligned}$$

Since $\Delta w = \sum_{j=1}^n w_{jj}$, we have $w_t \leqslant \Delta w$. Therefore, the maximum principle implies

$$|\nabla u|^2 \leqslant \max_{\overline{Q_T}} w = \max_{\partial_p Q_T} w.$$

Hence, it is enough to show $|\nabla u|^2 \leqslant \max |\nabla g|^2$ on $\partial\Omega$.

On the other hand, since $\partial\Omega$ is a level set of u, we have $|\nabla u| = |u_\nu|$ on $\partial\Omega$ where $u_\nu = \langle \nabla u, \nu\rangle$. Thus, we only need to show $|u_\nu| \leqslant \max |\nabla g|$ on $\partial\Omega$.

Now, we pick any point $x_0 \in \partial\Omega$ and show $|u_\nu| \leqslant \max |\nabla g|$ at the point. Then, by rotating and translating Ω, we can assume $x_0 = 0$ and $\nu = -e_1$ without loss of generality. Next, we define a *barrier* $v(\vec{x}, t)$ by

$$v(\vec{x}, t) = Ax_1, \qquad \text{where } A = \max |\nabla g|.$$

Then, we can observe $u \leqslant v$ on $\partial_p Q_T$. (Here, we used the convexity of Ω. Draw a picture!!) In addition, $v_t = \Delta v = 0$. By the maximum principle, the following holds in $\overline{Q_T}$.

$$u(\vec{x}, t) \leqslant Ax_1.$$

In the same manner, we can show $u \geqslant -Ax_1$. Thus, $|u_\nu| = |u_1| \leqslant A$ at $\vec{x} = 0 \in \partial\Omega$.

□

One maybe curious about general Dirichlet boundary data. Suppose $u(\vec{x}, t) = \varphi(\vec{x}, t)$ on $\partial\Omega$. Given $\vec{x}_0 \in \partial\Omega$, we consider a tangential direction τ of $\partial\Omega$ at x_0. Then, we choose a vector-valued function $\gamma : (-\epsilon, \epsilon) \to \partial\Omega \subset \mathbb{R}^n$ satisfying $\gamma(0) = \vec{x}_0$, $\gamma'(0) = \tau$. Then, given t_0 we define $g(s) = u(\gamma, t_0) = \varphi(\gamma(s), t_0)$, and differentiate to obtain

$$g'(0) = \langle \nabla u(\vec{x}_0, t_0), \gamma'(0)\rangle = u_\tau(\vec{x}_0, t_0) = \langle \nabla\varphi(\vec{x}_0, t_0), \gamma'(0)\rangle = \varphi_\tau(\vec{x}_0, t_0).$$

Namely, $|u_\tau| = |\varphi_\tau| \leqslant |\nabla\varphi|$ on the boundary.